ONE SELF-SIMILAR SOLUTION TO THE MAGNETOHYDRODYNAMIC EQUATIONS

(OB ODNOM AVTOMODEL'NOM RESHENII URAVNENII MAGNITNOI GIDRODINAMIKI)

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This paper deals with steady inviscid supersonic flow round a wedge, on the surface of which there are surface currents which flow perpendicularly to the air flow, the square of the intensity of these currents being inversely proportional, to the distance from the vertex of the wedge. The problem is posed in the same manner as in [1]. The gas is assumed to be nonconducting in front of the shock wave and it is finitely conducting in the region of the disturbed current, whilst the transition through the shock wave is described by the same relationships as those which would hold without the presence of the magnetic field. Within these assumptions there does exist a self-similar solution of the magnetohydrodynamic equations which describes flow close to the vertex of the wedge and, it is indeed a generalization of the well known accurate solution of the problem of flow of a supersonic stream round a wedge in the absence of a magnetic field.

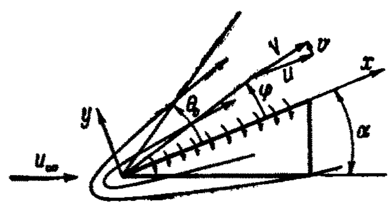

Fig. 1.

For a given magnetic field intensity there exist two solutions I of the boundary value problem obtained from the appropriate system of three ordinary differential equations. When the intensity of the magnetic field tends to zero these solutions go to the well known nonmagnetic solutions to the problem of flow round a wedge with both weak and strong shock and the velocity of the gas on the wedge surface is not zero. Additionally, there exist infinite numbers of solutions II for which the shock angle takes any value within the interval between the values obtained from solution I.

Solution II is characterized by zero velocity on the wedge surface and it has no analog in normal hydrodynamics. The requirement for a continuous relationship between the solution and the intensity of the magnetic field leads one to the conclusion that in practice the solution of type I which corresponds to weak shock will be realized. It is demonstrated for this solution that there is a continuous transition from flow with an attached shock wave in the absence of a field to flow with a detached shock wave with a rather strong field.

Some examples are given which have been worked out on the high speed M-20 computer.

1. Consider a plain inviscid supersonic stream flowing steadily over the top surface of a wedge which makes an angle α with the direction of the undisturbed stream velocity vector $\mathbf{U}_\infty$. Cartesian coordinates xy are directed as shown in Fig. 1.

It is assumed that surface currents whose density is given by

$$j = -Ax^{-1/2}$$

where A is a dimensional constant, pass normally to the plane of flow on the surface. The wedge has sides of length L.

The components h_x and h_y of the magnetic field vector $\mathbf{h}$ induced by the given distribution j are given by the relations

$$h_x = \frac{\sqrt{2}A}{c\sqrt{x}}\left(\frac{\pi}{2} + \tan^{-1}\Lambda\right) f(\eta) - \frac{A \ln z}{c\sqrt{2x}}\varphi(\eta) \tag{1.1}$$

$$h_y = \frac{\sqrt{2}A}{c\sqrt{x}}\left(\frac{\pi}{2} + \tan^{-1}\Lambda\right) \varphi(\eta) - \frac{A \ln z}{c\sqrt{2x}} f(\eta) \tag{1.2}$$

$$z = \left|\frac{L + \sqrt{2L}\sqrt{(x^2+y^2)^{1/2} + x} + \sqrt{x^2+y^2}}{L - \sqrt{2L}\sqrt{(x^2+y^2)^{1/2} + x} + \sqrt{x^2+y^2}}\right|, \quad \Lambda = \frac{L - (x^2+y^2)^{1/2}}{\sqrt{2L}\sqrt{(x^2+y^2)^{1/2} - x}} \tag{1.3}$$

$$f(\eta) = \sqrt{\frac{(1+\eta^2)^{1/2}+1}{1+\eta^2}}, \qquad \varphi(\eta) = \sqrt{\frac{(1+\eta^2)^{1/2}-1}{1+\eta^2}}, \qquad \eta = \frac{y}{x} \tag{1.4}$$

(c is the velocity of light in vacuum).

The conductivity σ is assumed to be zero in front of the shock wave, whilst behind the shock wave the gas conductivity depends on thermal ionization and it is finite so that the magnetic Reynolds number is given by $R_m = 4\pi\sigma U_\infty L/c^2 \leqslant 1$. The problem is posed in the same way as in [1], (the difference between the two cases involves the choice of magnetic field which is assumed in [1] to be constant and directed normally to the surface). The solution is constructed for flows with an attached shock wave so that the stream along the upper surface of the wedge may be considered to be independent of that flowing past the lower surface. Induced magnetic fields which could cause these flow fields to interact, may be neglected, as will be seen from what follows.

The solution is constructed in the neighborhood of the vertex of the wedge in the region $(x^2 + y^2)^{1/2} \leqslant l \ll L$, within which, from Equations (1.1) to (1.4) (with relative error of the order of $\sqrt{(l/L)}$, we have, for the given magnetic field

$$h_x = \frac{\sqrt{2\pi} A}{c\sqrt{x}} f(\eta), \qquad h_y = \frac{\sqrt{2\pi} A}{c\sqrt{x}} \varphi(\eta) \tag{1.5}$$

Now represent the resultant magnetic field vector in the form $\mathbf{h} + \mathbf{h}'$, where $\mathbf{h}'$ is the induced magnetic field. For $\mathbf{h}'$, from the expression for Ohm's law, taking into consideration the absence of electric field, we arrive at the relationship

$$\operatorname{rot} \mathbf{h}' = R_m \mathbf{V} \times (\mathbf{h} + \mathbf{h}')$$

where $\mathbf{V}$ is the velocity vector made dimensionless by division through U_∞. x and y are referred to L. If we use Newton potentials it is possible to show that h' is everywhere finite, and in the region of the disturbed stream is of order $R_m A/c\sqrt{L}$. Thus within the region we are discussing close to the vertex of the wedge the ratio h'/h of the induced and applied field is of order $\sqrt{(l/L)}$. It follows that both the finiteness of L and the induced magnetic field may be neglected with an error of order $\sqrt{(l/L)}$.

The equations of magnetohydrodynamics behind a shock wave in dimensional variables (for simplicity we assume a perfect gas) may be written

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\sigma h_y}{\rho}(vh_x - uh_y)$$
$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\sigma h_x}{\rho}(uh_y - vh_x) \tag{1.6}$$
$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \qquad \frac{u^2 + v^2}{2} + \frac{\varkappa}{\varkappa - 1}\frac{p}{\rho} = \frac{U_\infty{}^2}{2}\left[1 + \frac{2}{(\varkappa - 1) M_\infty^2}\right]$$

In these expressions u and v are components of the velocity vector along the x, y axes respectively, ρ is the density, p the pressure, σ the specific electrical conductivity, κ the adiabatic index, M_∞ the Mach number of the undisturbed stream. It will be assumed that σ is constant over the whole region of the undisturbed stream and M_∞ is infinite. It should be noted that the assumptions (σ = const, $M_\infty = \infty$, for a perfect gas) are not essential for the existence of a self-similar solution and are only made to simplify the investigation.

Assuming that the magnetic field is given by Equations (1.5) whilst the boundary conditions for the hydrodynamic quantities reduce to the condition of impermeability on the body, i.e. no flow through the surface, and to the usual nonmagnetic relationships on the shock wave [1], and making use of dimensional analysis [2], it is easy to convince oneself of the existence of a self-similar solution of Equation (1.6) in which the velocity vector, pressure and density are constant along rays which pass through the vertex of the wedge, i.e. they depend on $\eta = \tan\varphi = y/x$, where φ is the angle of inclination of a ray to the x-axis. The angle θ_0 of an oblique shock which emanates from the vertex of the wedge (Fig. 1) is determined in the process of the solution.

Now eliminate the pressure from Equation (1.6) using the foregoing equation and relate the velocity and density to their values in the undisturbed stream, retaining for the nondimensional values the same notations as for the dimensional ones. The result of going over to the independent variable η is to arrive at a system of three ordinary differential equations (strokes denote differentiation with respect to η)

$$u' = \frac{q\gamma P_1}{\rho\beta\Delta}, \qquad v' = \frac{q\gamma P_2}{\rho\beta\Delta}, \qquad \rho' = \frac{q\gamma P_3}{\beta\Delta} \tag{1.7}$$

Here

$$\gamma = fv - \varphi u, \qquad \beta = v - \eta u, \qquad \Delta = \beta^2 - (1 + \eta^2)\, a^2$$

$$P_1 = \mu\,(a^2 - v\beta) + \varkappa\beta\gamma\eta \qquad \left(a^2 = \frac{\varkappa - 1}{2}(1 - u^2 - v^2)\right)$$

$$P_2 = \mu\,(a^2\eta + u\beta) - \varkappa\beta\gamma, \qquad P_3 = -\mu\,(u + v\eta) + \varkappa\gamma\,(1 + \eta^2) \tag{1.8}$$

$$q = \frac{2\sigma A^2\pi^2}{c^4\rho_\infty U_\infty} \qquad \left(\mu = f\eta - \varphi = \sqrt{(1 + \eta^2)^{1/2} - 1}\,\right)$$

where $f(\eta)$ and $\varphi(\eta)$ are determined according to (1.4), the intensity of the magnetic field is represented by the dimensionless parameter $q > 0$, where ρ_∞ is the density of the undisturbed stream.

It should be observed that there exists a similar self-similar solution for the case of an axially symmetric flow past a cone with azimuthal surface currents whose densities (as in the plane case) are inversely proportional to the square root of the distance from the vertex of the cone.

2. The system (1.7) is studied in the velocity hodograph where the motion of a representative point $V(u, v)$ is followed as η changes. The solution is defined within a circle C_1 of unit radius the square of the velocity of sound being $a^2 > 0$.

On the surface uv families (1.8) of curves β, γ, Δ, P_1, P_2, P_3 are given, each of which depends on the parameter η. (Here and below the curves β, γ, etc. denote curves which are obtained from the equations $\beta = 0$, $\gamma = 0$ etc.) On curves β and Δ the derivatives u', v' and ρ' go to infinity, i.e. these lines become singular. On the rest of the curves the derivatives u', v' and ρ' vanish according to (1.7). These curves are depicted in Fig. 2 where they are represented for a fixed value of η. The straight lines β, γ and P_3 go through the origin (the straight line P_3) has not been taken to C_1 so as not to confuse the drawing). Their angular coefficients are respectively k_β, k_γ, k_3 and are expressed in the following way

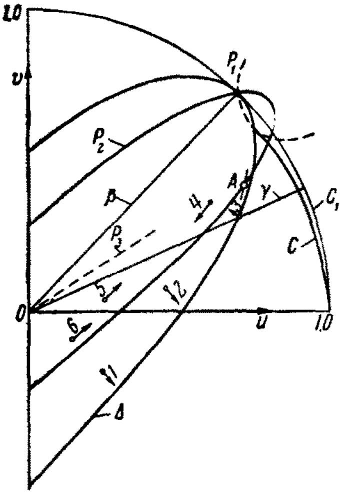

Fig. 2.

$$k_\beta = \eta = \tan\varphi, \quad k_\gamma = \frac{\varphi}{f} = \frac{(1+\eta^2)^{1/2}-1}{\eta}$$

$$k_\beta > k_3 = k_\gamma \frac{\varkappa(1+\eta^2)^{1/2}+1}{(\varkappa-1)(1+\eta^2)^{1/2}+1} > k_\gamma \qquad (2.1)$$

The straight lines β and γ have a simple physical explanation; the line β divides the regions where the angle of inclination of the velocity vector with the x-axis is less ($\beta < 0$) and greater ($\beta > 0$) than the angle of inclination φ of a given ray in the physical plane. Similarly γ divides the region where the angle between the velocity vector and the x-axis is less ($\gamma < 0$) and greater ($\gamma > 0$) than the angle between the magnetic field vector (1.5) and the x-axis. The curves Δ and P_2 are ellipses, P_1 is a hyperbola. The curve Δ is well known from Busemann's theory of characteristics (see, for instance, [3]); its major and minor axes are equal respectively to unity and to $(\kappa-1)^{1/2}/(\kappa+1)^{1/2}$.

In this problem the major axis of ellipse Δ is directed along line β, i.e. it makes an angle φ with the u-axis equal to the angle between the corresponding ray in the physical plane and the x-axis. If point V, always located below the straight line β, as will be seen later, lies within, on or outside the ellipse Δ, it shows that the angle of inclination of the characteristic of the first family with the x-axis is respectively greater than, equal to or less than φ. This follows from the circumstance that the characteristic equation in the physical plane obtained from (1.6) has the same form as in normal hydrodynamics.

On the ellipse P_2 when η changes from zero to ∞ the major axis turns counter-clockwise and decreases from $2\kappa/(3\kappa + 1)^{1/2}$ to 1 whilst the minor axis increases from 0 to $(\kappa - 1)^{1/2}/(\kappa + 1)^{1/2}$. When $\eta \to \infty$ the ellipse P_2 coincides with Δ. The ellipse P_2 intersects C_1 twice; on line β and on the ray whose angular coefficient is k_2.

The hyperbola P_1 intersects the circle C_1 on line β and on a ray whose angular coefficient is k_1 where,

$$k_1 = \frac{\varkappa\eta}{(\varkappa - 1)(1+\eta^2)^{1/2} + \varkappa} < k_2 = \frac{[(1+\eta^2)^{1/2} - 1]\,[(1+\eta^2)^{1/2} + \varkappa]}{\varkappa\eta} \qquad (2.2)$$

When $\eta = 0$ the vertex of the hyperbola lies at the point $u = 1$, $v = 0$ whilst its axis coincides with the u-axis

$$u^2 - \frac{3\varkappa - 1}{\varkappa - 1} v^2 = 1 \qquad (2.3)$$

When η increases the axis of the hyperbola moves counter-clockwise whilst the vertex approaches the origin. When $\eta \to \infty$ the hyperbola P_1 decays into a pair of straight lines, $u = 0$, $v = \kappa u/(\kappa - 1)$. On Fig. 3 β, P_1 and P_2 are shown for $\eta = 0.2$ (lightly dotted), $\eta = 1$ (heavy dots) and $\eta = 10$ (dots and dashes).* Here the envelope of hyperbolas P_1 is also shown (line C on Figs. 2 and 3). It can be shown that the strophoid** (circle C_2 in Fig. 3), constructed for any positive value of the angle α between the undisturbed velocity and the x-axis and giving the condition of the stream behind the shock wave, always lies within the envelope and never intersects it when the angle θ_0 varies (Fig. 1). The only possible exception is point A (Fig. 3) in which when $\alpha = 0$ the strophoid touches the envelope from within. The equations of envelope and of strophoid (when $\alpha = 0$) near point A may be written down respectively thus

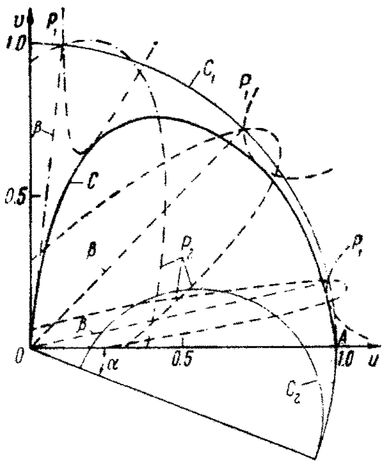

Fig. 3.

* Both here and below all constructions and calculations relate to $\kappa = 1.4$.

** It is known that when $M_\infty = \infty$ the strophoid decays into a circle.

$$u = 1 - \frac{3\varkappa - 1}{4\varkappa} v^2 + O(v^4)$$
$$u = 1 - \frac{\varkappa + 1}{2} v^2 + O(v^4) \tag{2.4}$$

3. Quantities which lie immediately behind the shock wave will be denoted by suffix 0 everywhere. As evidently behind the wave the angle between the velocity vector and the x-axis is less than θ_0 we have $\beta_0 < 0$, i.e. the representative point V_0 immediately behind the wave lies below the straight line β. This point moreover lies within Δ, for with supersonic velocity behind the wave the angle of inclination of the characteristic of the first family exceeds θ_0 and with subsonic velocity V it always lies within the Busemann ellipse. Thus behind the shock wave the point V_0 may lie in one of six regions (Fig. 2). Within each of the regions the derivative dv/du along the integral curve (1.7) does not change sign. The direction of motion of point V with decrease in η, i.e. with motion of the shock wave to the body is shown in Fig. 2 by means of an arrow. Note that P_3 is not a boundary of the region. Besides in accordance with Section 2, in region 4 point V_0 lies outside the hyperbola P_1. We will now prove several theorems about the motion of V.

a) From Equations (1.7) and (1.8) for the quantities γ along the integral curve we have on line γ

$$\left(\frac{d\gamma}{d\eta}\right)_{\gamma=0} = -\frac{v}{(1+\eta^2)\left[(1+\eta^2)^{1/2}-1\right]^{1/2}} < 0 \tag{3.1}$$

under the assumption $v > 0$ and $\eta > 0$. It follows from this that in the case of motion from shock wave to body, i.e. decrease in η, the quantity γ cannot change sign from positive to negative (the transition in the reverse direction may take place).

b) The integral curve does not reach line β for $\eta > 0$. We are going to give the proof by proving the impossibility of its opposite. Suppose that at some point (we will characterize it by suffix $_*$) the quantity β becomes zero. We will write down the auxiliary equations which follow from Equations (1.7), and (1.8):

$$\frac{d\ln\rho}{du} = \frac{P_3}{P_1}, \qquad \left(\frac{q\gamma P_3}{\rho\Delta} + u\right)\frac{du}{d\beta} = -\frac{q\gamma P_1}{\rho\beta\Delta} \tag{3.2}$$

We choose on the given integral curve in the neighborhood of point η_* some other point which we will denote by $_{**}$. It follows from Section 2 that over the interval $\eta_*\eta_{**}$ the quantities P_1, P_2, P_3 do not vanish anywhere and they do not change sign.

From the first equation (3.2) we have for ρ

$$\ln \frac{\rho}{\rho_{**}} = \int\limits_{\eta_{**}}^{\eta} \frac{P_3}{P_1} du \tag{3.3}$$

where the integral on the right-hand side of (3.3) is already known to converge, i.e. ρ is a finite quantity within the closed interval $\eta_* \eta_{**}$. Furthermore, from the second equation (3.2) we have

$$\int\limits_{u_{**}}^{u} \frac{q\gamma P_3 + \rho u \Delta}{q\gamma P_1} du = \ln \frac{\beta_{**}}{\beta} \tag{3.4}$$

If we let u tend to u_* we arrive at a contradiction; the left-hand side of Equation (3.4) remains finite whilst the right-hand side grows without limit for $\beta_* = 0$, and this proves the theorem.

c) The integral curve cannot intersect the line P_2 when point V moves from within the ellipse. Suppose on the contrary, that at point A there has been some intersection of the integral curve with ellipse P_2 (Fig.2). As the angular coefficient of the tangent to P_2 is positive in the given region, then along the integral curve $dv/du > 0$ at point A. On the other hand from the first and second equations (1.7) it follows that $dv/du = 0$ at this point. The contradiction in fact confirms the statement made above.

d) In region 4 the integral curve does not intersect the line P_1. Such an intersection evidently could take place at high enough values of η when the vertex of the hyperbolas is close to the origin. The proof is similar to the one above and rests on the first two equations (1.7).

e) The density ρ differs from zero and infinity everywhere, except at the two points of intersection of lines P_1 and P_2. This follows from Equation (3.3) because the expression in the integral sign on the right-hand sign is finite. If the integral curve intersects line P_1 at a point within the interval $\eta\eta_{**}$, then instead of (3.3) in the neighborhood of this point we make use of the similar equation

$$\ln \frac{\rho}{\rho_{**}} = \int\limits_{\eta_{**}}^{\eta} \frac{P_3}{P_2} dv \tag{3.5}$$

f) The ellipse Δ is the limiting line for the solutions (1.7). Actually suppose at some point represented by * lying at the boundary of regions *1* or *2* (Fig. 2), the integral curve intersects the line Δ. In the neighborhood of this point we have $\Delta = (u - u_*)\omega$, where $\omega = \omega(u, v, \eta)$, $\omega_* > 0$. The first equation (1.7) can be written thus

$$\frac{du}{d\eta} = \frac{\Omega}{u - u_*}, \qquad \Omega = \frac{q\gamma P_1}{\rho\beta\omega}, \qquad \Omega_* > 0 \tag{3.6}$$

The solution (3.6) in the neighborhood of the singular point has the form

$$|u - u_*| = \sqrt{2\Omega_* (\eta - \eta_*)}$$

i.e. it cannot be continued into the region $\eta < \eta_*$.

4. Consider now the behavior of the integral curves, when V lies in region 4. According to the theorems (Section 3, a, b, c) when $\eta > 0$ the integral curve cannot go outside region 4. When η decreases the component velocities u and v decrease monotonically. It will be seen (Section 7) that when η tends to zero when region 4 decays into a section of the u-axis it is not possible to obtain a solution for which the representative point tends to some point with coordinates $v = 0$, $u = u_1 \neq 0$ within region 4. Thus in region 4 one should search for a solution of (1.7) in which $\lim u = 0$ and $\lim v = 0$ for $\eta \to 0$. Write down now the system (1.7) for η close to zero in a simplified form expanding $f(\eta)$ and $\varphi(\eta)$ in series, and retaining the first terms, bearing in mind also how small u and v are.

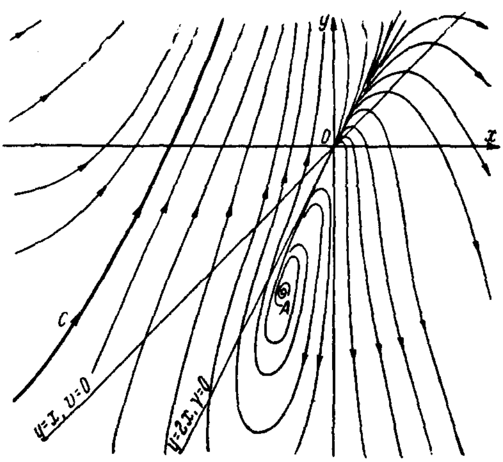

Fig. 4.

We obtain

$$u' = -\frac{q(v - 0.5\,\eta u)\,\eta}{\rho(v - \eta u)}$$

$$v' = \eta u', \quad \rho' = 0 \tag{4.1}$$

Introducing β as an unknown function, we obtain the equations

$$\beta\beta'' + \frac{\delta}{2}\beta'\eta^2 - \delta\beta\eta = 0, \quad u = -\beta', \quad v = \beta + \eta u$$

$$\rho = \rho_1 = \text{const}, \quad \delta = \frac{q}{\rho_1} \tag{4.2}$$

The first equation (4.2) for β allows for a group transformation and can be reduced to an equation of the first order in the quantities $x = \beta/\eta^3$, $y = \beta'/\eta^2$.

$$\frac{dy}{dx} = \frac{\delta(0{,}5\,y - x) + 2xy}{x(3x - y)}, \qquad \frac{d \ln \eta}{dx} = \frac{1}{y - 3x} \tag{4.3}$$

Equation (4.3), whose integral curves are shown in Fig. 4 (the arrows show the directions of increasing η), has two singular points; the point O ($x = 0$, $y = 0$, saddle and node running together) and A ($x = -\delta/12$, $y = -\delta/4$, i.e. focus). Region 4 on this figure lies in the third quadrant within the angle made by the straight lines $y = 2x$ (corresponding to our approximation $\gamma = 0$) and $x = 0$ ($\beta = 0$). In accordance with the theorems proved, the integral curves which enter this region, tend toward point A. The latter gives an analytical solution (4.3) which satisfies the condition of impermeability (i.e. not flowing through)

$$u = \frac{\delta}{4}\eta^2, \qquad v = \frac{\delta}{6}\eta^3, \qquad \rho = \rho_1 = \text{const} \tag{4.4}$$

To find the other solutions close to the singular point we transform the first equation (4.3), assuming $x = -\delta/12 + \xi$, $y = -\delta/4 + \zeta$, where ξ and ζ are small quantities. If we integrate the equations so obtained we arrive at

$$\sqrt{(\zeta - 3.5\,\xi)^2 + 5.75\,\xi^2}\, e^{\varphi/\sqrt{23}} = \text{const}, \qquad \varphi = \tan^{-1} \frac{2(\zeta/\xi - 3.5)}{\sqrt{23}} \tag{4.5}$$

From (4.5) and from the second equation (4.3) we obtain (C_1 and C_2 are arbitrary constants)

$$\frac{\beta}{\eta^3} + \frac{\delta}{12} = \frac{\nu C_1 \sqrt{\eta}}{\sqrt{1 + \tan^2 \chi}}, \qquad \chi = \sqrt{23} \ln \frac{C_2}{\sqrt{\eta}}, \qquad \nu = \frac{\cos \chi}{|\cos \chi|} \tag{4.6}$$

If use is made of (4.2) and (4.6), expressions for u, v, ρ can be found which, if substituted into the basic equations (1.7), show that solution (4.6) of Equation (4.1) really does give the main terms of the asymptotic expansions of the solutions near the singularities. The following terms of the expansion may be obtained from (1.7). When $C_1 = 0$, when solution (4.6) transforms to (4.4), it is possible to arrive at a solution in power series form

$$\begin{aligned} u &= \frac{\delta}{4}\eta^2 - \frac{15\delta}{64}\eta^4 + \ldots \\ v &= \frac{\delta}{6}\eta^3 - \frac{3\delta}{16}\eta^5 + \ldots \\ \rho &= \rho_1 \left(1 - \frac{3 - \varkappa}{\varkappa - 1}\frac{\delta^2\eta^4}{48} + \ldots\right. \end{aligned} \tag{4.7}$$

In a similar manner it is possible to obtain solutions for C_1 not equal to 0 when they are not analytic functions, when $\eta = 0$.

Thus the general solution in region *4* which satisfies the condition of impermeability for $\eta = 0$ depends on the arbitrary constant quantities $\rho_1 > 0$, C_1 and C_2. For this solution not only v but also u vanish on the surface of the body. Making use of two conditions on a shock wave which connect the three unknown functions u, v, ρ, it is possible to exclude the dependence on two parameters, and so represent the solution as a set of relations in terms of one parameter which may for instance be taken to be the angle of inclination of the shock wave θ_0 * with q fixed or q with θ_0 fixed. Under these conditions q may vary between 0 and ∞ (Section 5 deals with the case where $q = \infty$, when solution (4.7) is inapplicable). Thus the solution imposed by the boundary problem in region *4* turns out to be positive or negative (multi-signed). On Fig. 5 a calculated example is shown for the case $\alpha = 0$, $\theta_0 = 45^\circ$, $q = 20$ and $q = 30$.

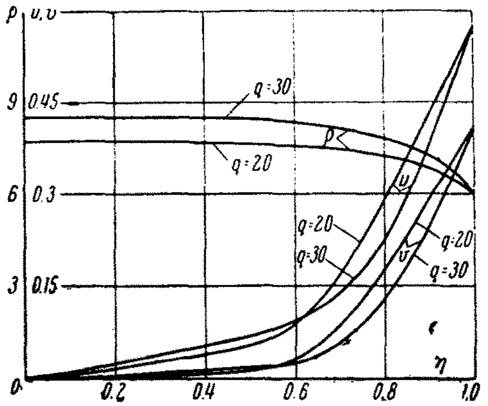

Fig. 5.

The solution which corresponds to zero velocity at the wall will be called solution II.

5. Evidently when V lies in region *3* (Fig. 2), when η is decreased, a transition takes place $3 \to 4 \to 0$ (this will denote a transition from one region to another and the tendency of the representative point to approach the origin for $\eta \to 0$), i.e. the solution in region *3* is also of single sign.

6. To complete the investigation of the solutions in regions *4* and *3* and for what follows it is essential to know the behavior of the integral curves in the limiting case $q = \infty$. To do this we transforms (1.7) into the following form

$$\frac{dv}{du} = \frac{P_2}{P_1}, \qquad \frac{d\ln\rho}{du} = \frac{P_3}{P_1}, \qquad \frac{d\eta}{du} = \frac{\rho 3\Delta}{q\gamma P_1} \tag{6.1}$$

If we make q tend to ∞ and assume $P_1 \neq 0$ we can identify two cases.

* Obviously θ_0 is such that V_0 lies within region *4*.

(a) Case for $\gamma \neq 0$

$$\frac{dv}{du} = \frac{P_2}{P_1}, \qquad \frac{d\ln\rho}{du} = \frac{P_3}{P_1}, \qquad \eta = \eta_0 = \text{const} \tag{6.2}$$

(b) Case for $\gamma = 0$. Substituting into the first and second of Equations (6.1) $\eta = 2uv/(u^2 - v^2)$ from condition $\gamma = 0$ and making use of (1.4) and (1.8) we arrive at

$$\frac{dv}{du} = \frac{2uv\,[1 - \varkappa(\varkappa-1)^{-1}(u^2+v^2)]}{u^2 - v^2 + (u^2+v^2)\,[(\varkappa+1)(\varkappa-1)^{-1}v^2 - u^2]} \tag{6.3}$$

$$\frac{d\ln\rho}{du} = -\frac{2u\,(u^2+v^2)}{(\varkappa-1)\,\{u^2 - v^2 + (u^2+v^2)\,[(\varkappa+1)(\varkappa-1)^{-1}v^2 - u^2]\}} \tag{6.4}$$

In cases (a) and (b) the equations in the hodograph plane are closed. The integral curves for $\gamma \neq 0$ are described by the first equation (6.2) up to the point when the integral curve does not intersect line $\gamma = 0$, after which (6.3) is used. If such an intersection does not take place (then the integral curve intersects line Δ, which indicates the lack of a solution with the accepted initial conditions (Section 3, e). When undergoing the transitions $4 \to 0$ or $3 \to 4 \to 0$, the representative point V after the intersection between the integral curve and line γ, moves along the "upper edge" of the cut $\gamma = 0$, tending to zero at the origin with decreasing η, in the neighborhood of the origin the main terms of the expansion of solution (6.3) and (6.4) take the following form

$$v = Au^2, \qquad \rho = \rho_1\Big(1 - \frac{u^2}{\varkappa - 1}\Big), \qquad \eta = 2Au, \qquad A = \text{const} \tag{6.5}$$

distinct from the corresponding solution (4.7) with finite q.

7. Consider now the solution in region 2. For small q, when point V hardly moves, when η decreases and the neighborhood of line β is not yet approached, the following transitions take place: $2 \to 5 \to 4 \to 0$ or $2 \to 3 \to 4 \to 0$.

According to the analysis, the point of the intersection S between lines γ and Δ is a moving node for Equation (1.7). With fixed q and decreasing η the representative point moves away from S; if q is large enough, the integral curve in region 2 intersects with line Δ, i.e. the solution evinces a limiting line. If q decreases the integral curve intersects line Δ for smaller values of η.

There exists a value $q = q_*$ when the integral curve wholly lies in region 2 whilst with η tending to zero, $v \to 0$ and $u \to u_1 \neq 0$ (for $q < q_*$ the solution II is obtained). Now examine the solution when the integral curve approaches the u-axis for $\eta \to 0$. To do this we simplify Equation

(1.7) making the assumption that v and η are small quantities. As a result we arrive at

$$u' = -\frac{q\,(v - 0.5\,\eta u_1)\,\eta}{\rho\,(v - \eta u_1)} \tag{7.1}$$

$$v' = -\frac{q\,(v - 0.5\,\eta u_1)}{\rho\,(v - \eta u_1)\,(1 - u_1^2)}\left[\left(1 - \frac{3\varkappa + 1}{\varkappa - 1}\,u_1^2\right)\eta^2 + \frac{2\,(3\varkappa + 1)}{\varkappa - 1}\,u_1 v \eta - \frac{4\varkappa v^2}{\varkappa - 1}\right] \tag{7.2}$$

$$\rho' = \frac{2q\,(v - 0.5\,\eta u_1)\,[(\varkappa + 1)\,\eta u_1 + 2\varkappa v]}{(\varkappa - 1)\,(v - \eta u_1)\,(1 - u_1^2)} \tag{7.3}$$

Equations (7.1) to (7.3) have an asymptotic solution when η tends to zero and they satisfy the conditions of impermeability, the main terms of which can be written

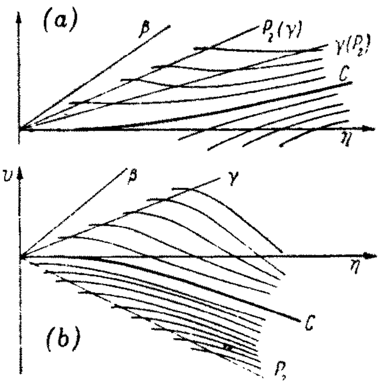

Fig. 6.

$$u = u_1 - \frac{q\eta^2}{4\rho_1},$$

$$v = \frac{q\eta^3}{6\rho_1\,(1 - u_1^2)}\left[\frac{3\varkappa + 1}{\varkappa - 1}\,u_1^2 - 1\right]$$

$$\rho = \rho_1\left[1 + \frac{q\,(\varkappa + 1)\,u_1\eta^2}{\rho_1\,(\varkappa - 1)\,(1 - u_1^2)}\right] \tag{7.4}$$

A solution in region *2* can take place for $u_1^2 > (\kappa - 1)/(3\kappa + 1)$, when the integral curves approach the u-axis for $\eta \to 0$ from above, and this means that $2 \to u_+$. When $u_1^2 < (\kappa - 1)/(3\kappa + 1)$ the integral curves can approach the u-axis for $\eta \to 0$ from below from region *6* and this will be denoted by $6 \to u_-$.

In order to investigate the singularity on the u-axis put $\rho = \rho_1$ in Equation (7.2). The integral curves of this equation in the $v\eta$ plane are shown in Fig. 6. In this figure *a* corresponds to the case $u_1^2 > (\kappa - 1)/(3\kappa + 1)$ and 6*b* to the case for $u_1^2 < (\kappa - 1)/(3\kappa + 1)$. Curves C correspond to solution (7.4) which depends on two parameters. If $1 > u_1 > \sqrt{[(\kappa - 1)/\kappa]}$ in Fig. 6, *a* line P_2 lies* above γ and curve C separates the integral curves $2 \to 3 \to 4 \to 0$, which go above C from the curves $2 \to 1$. If $\sqrt{[(\kappa - 1)/\kappa]} > u_1 > \sqrt{[(\kappa - 1)/(3\kappa + 1)]}$ line P_2 lies below γ and curve C separates the curves $2 \to 5 \to 4 \to 0$ from the curves $2 \to 1$. In a similar manner in Fig. 6*b*,** the curve C separates the curve $6 \to 1$ from the

* In this approximation with coordinates $v\eta$ the line P_2 and also γ will be straight.

** Using other coordinates curve C is shown on Fig. 4 for the case $1 - u_1^2 \approx 1$.

curves $6 \to 5 \to 4 \to 0$. The singular point on the u-axis therefore is similar to a saddle and curves C enter it from definite directions.

If we make use of Equations (1.7) solution (7.4) can be made more accurate by finding the following terms in the expansion in series in η. For instance the expression for v in $u_1^2 = (\kappa - 1)/(3\kappa + 1)$ takes the following forms:

$$v = \frac{q}{10\sqrt{2}(\kappa+1)\rho_1}\left[\kappa + \frac{q}{2\rho_1}\sqrt{\frac{(3\kappa+1)^3}{\kappa-1}}\right]\eta^5 \tag{7.5}$$

Owing to the fact that solution (7.4) depends on two parameters, whilst on the shock wave there are two conditions which connect u, v, ρ, in region 2 to each value of q there corresponds one unique solution I which satisfies the condition of impermeability as distinct from solution II dealt with in Section 4.

When q_* tends to zero the angle of inclination of the shock wave tends to its "non-magnetic" value θ_{00}. The representative point behind the shock wave lies on the u-axis and does not move when η decreases to zero, i.e. the solution in region 2 tends to the well known exact solution of the wedge problem (weak shock solution).

Figure 7 illustrates an example in which calculations are done for $\alpha = 30^\circ$, $\theta_0 = 11^\circ.3$ ($\eta_0 = 0.2$). The quantity $q_* = 18.056$ was so chosen that the solution for $\eta \to 0$ took the form (7.4).

Solution (7.4) is valid for finite q and $u_1 \neq 1$. When $q = \infty$ the conditions of impermeability for $u_1 = 1$ can be satisfied. To do this it is necessary to see that V_0 lies in region 2 on the "lower edge" of the cut $\gamma = 0$. When η is reduced, point V will move toward the circle C_1 (Fig.2). A solution to Equations (6.3), (6.4), satisfying the condition of impermeability has the following form

$$\begin{gathered} u = 1 - Cv^{\kappa-1} + \ldots \\ \eta = 2v\,(1 + Cv^{\kappa-1} + \ldots) \\ \rho = Bv\,(1 + 2Cv^{\kappa-1} + \ldots) \\ C = \text{const} > 0, \qquad B = \text{const} > 0 \end{gathered} \tag{7.6}$$

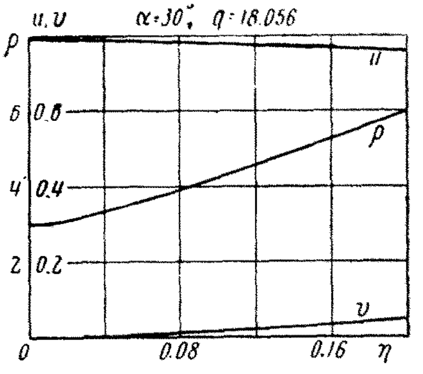

Fig. 7.

The solution (7.6) can be constructed for $\kappa \leqslant 3$. One can demonstrate that for $\kappa > 3$ the representative point behind the wave cannot be on the boundary of region 2 on the line γ. It is not out of place to note that for $q = \infty$ the density (and therefore pressure) on the wedge surface are both zero.

8. If we make use of the assumption that the gas in the shock wave is in a high state of compression [4] (this is equivalent, as is well known, to the assumption $\varepsilon = (\kappa - 1)/(\kappa + 1) \ll 1$), an approximate solution can be constructed in region 2 for values of q where the solution with magnetic field differs little from the corresponding nonmagnetic solution. Now if we make use of the relationships for the quantities on the shock wave (Fig. 1) when $\varepsilon \ll 1$

$$v_0 = \theta_0 (1 - \varepsilon) \cos\alpha - (\varepsilon + \theta_0^2) \sin\alpha, \quad u_0 = (1 - \theta_0^2) \cos\alpha - (1 - \varepsilon)\, \theta_0 \sin\alpha$$
$$\rho_0 = \varepsilon^{-1} \tag{8.1}$$

and from (7.4), we have with an error of magnitude ε^3

$$\theta_0 = \varepsilon \tan\alpha + \frac{\varepsilon^2 \tan\alpha}{\cos^2\alpha} + \frac{q\varepsilon^4 \sin\alpha\, [(3\varkappa + 1)\cos^2\alpha/(\varkappa - 1) - 1]}{6\cos^4\alpha}$$
$$u = \cos\alpha \left[1 - \varepsilon \tan^2\alpha + \frac{q\varepsilon}{4\cos\alpha}(\varepsilon^2 \tan^2\alpha - \eta^2)\right]$$
$$v = \frac{\varepsilon q \eta^3\, [(3\varkappa + 1)\cos^2\alpha/(\varkappa - 1) - 1]}{6\sin^2\alpha}, \quad \rho = \frac{1}{\varepsilon}\left[1 - \frac{q\,\varepsilon^3}{\cos\alpha}\left(1 - \frac{\eta^2}{\varepsilon^2 \tan^2\alpha}\right)\right] \tag{8.2}$$

Formulas (8.2) can be applied when $q\varepsilon \lesssim 1$, and this follows from (7.4).

9. When point V comes into region 1, the condition of impermeability cannot be satisfied and no solution to the boundary value problem exists.

10. Behavior of integral curves is now analyzed when point V_0 lies in region 5 or region 6.

When q is sufficiently small, when V hardly moves, when η is decreased a transition $5 \to 4 \to 0$ takes place, whilst in region 6 a solution to the problem does not exist. The following theorems can be proved and these are based on evaluations for the magnitude of dv/du for finite (1.7) and infinite (6.2) values of q.

a) If, when $q = \infty$, the integral curve emanating from some point A of region 5 (Fig. 8) intersects γ, then for any value of q the integral curve will intersect γ (the points of intersection are shown on Fig. 8 by circles and they come closer to A with decrease in q; the symbol ∞ relates to curve $q = \infty$). Thus for any values of q a transition $5 \to 4 \to 0$ takes place from point A.

b) If the integral curve (6.2) for $q = \infty$, emanating from some point C

in region 6, intersects γ, there exists some value $q_* > 0$ such that for all values $q > q_*$ the transition $6 \to 5 \to 4 \to 0$ is realized.

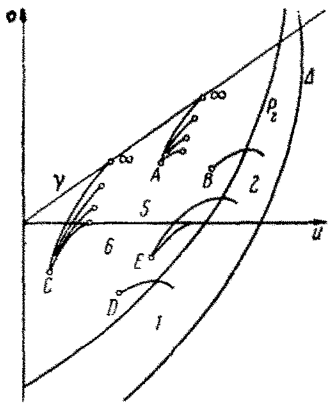

Fig. 8.

When $q = q_*$ a condition of impermeability in region 6 can be satisfied in accordance with (7.4) with $u_1^2 < (\kappa - 1)/(3\kappa + 1)$ and this corresponds to the transition $6 \to u_-$ (curve C on Fig. 6, b).

It is of interest to observe that, when $q \to 0$, a solution to the boundary value problem exists when the representative point immediately behind the shock wave lies on the u-axis. This solution corresponds to the "strong shock" usual in the normal hydrodynamics (Section 7).

c) If the integral curve (6.2) when $q = \infty$, in coming from some point V in region 5 (Fig. 8), intersects line P_2 and goes over into region 2, evidently a limiting line will appear in the solution; the integral curve will intersect Δ. Let us examine in this case how the solution is altered when we gradually increase q from zero. When q is small the transition $5 \to 4 \to 0$ takes place. Then for any value of q the straight line γ "overtakes" the representative point on line P_2. Further increase in q results in an intersection between the integral curve and the line γ which lies at the boundary of regions 2 and 3, and it moves downwards towards the u-axis. When this happens the transition $5 \to 2 \to 3 \to 4 \to 0$ takes place until finally, when $q = q_*$, solution $5 \to 2 \to u_+$ is no longer possible. When $q > q_*$ there is no solution to the boundary value problem (Section 7).

d) When V_0 lies at the boundary of region 5 on the lower edge of the cut γ the following cases may exist:

(1) dv/du determined by the conditions behind the shock wave are greater than k_γ (2.1). This case is similar to case (a). Evidently a transition $5 \to 4 \to 0$ will always take place.

(2) $dv/du < k_\gamma$, here as in (c) transitions $5 \to 4 \to 0$ are possible and also $5 \to 2 \to 3 \to 4 \to 0$. When $q = \infty$ the representative point moves along the lower edge of the cut γ in accordance with (6.3) whilst, when $\eta \to 0$ we arrive at solution (7.6).

e) If along the integral curve (6.2) with $q = \infty$ emanating from some point E in region 6 (Fig. 8), there takes place a transition $6 \to 5 \to 2$, then a limiting line will appear in the solution exactly as in case (b).

In this case over some range of variation $q(q_{*-} < q < q_{*+})$ the transitions $6 \to 5 \to 4 \to 0$ or, (for high values of q), $6 \to 5 \to 2 \to 3 \to 4 \to 0$ will take place. When $q = q_{*-}$ this solution $6 \to 5 \to 2 \to u_+$ will exist whilst when $q = q_{*+}$ solution will be $6 \to 5 \to 2 \to u_+$. A solution close to the u-axis has the form (7.4). Values of u on the surface of the wedge for these solutions (respectively u_{1-} and u_{1+}) satisfy the inequality $u_{1-}^2 < (\kappa - 1)/(3\kappa + 1) < u_{1+}^2$. Figure 9 illustrates a sample calculation for $\alpha = 30^\circ$, $\theta_0 = 54^\circ.5$ ($\eta_0 = 1.4$) when the representative point on the wave V_0 lies within region 6. The quantities $q_{*-} = 1.35$ and $q_{*+} = 4.0$ were chosen so that the solution for $\eta \to 0$ took the form (7.4).

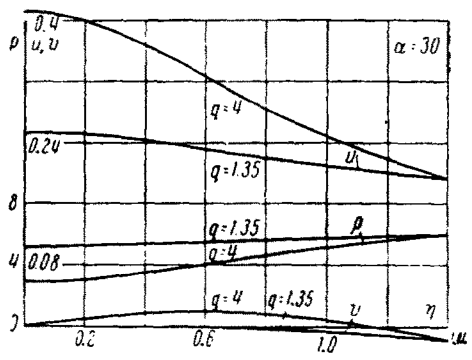

Fig. 9.

f) In region 6 there are points (D in Fig. 8) for which for any value of q no solution to the boundary value problem exists.

11. Now fix the angle α and, using the results of Sections 4 to 10 study the motion V_0 along the strophoid (Fig. 10a) when θ_0 varies. Points A and D describe the flow when $q = 0$, respectively behind a weak and strong shock, and they correspond to the angles α of directional change of the stream. If the angle of inclination of the shock wave becomes less than θ_{00}, i.e. its nonmagnetic value for weak shock, the representative point appears in region 1 (Section 9), where no solution exists.

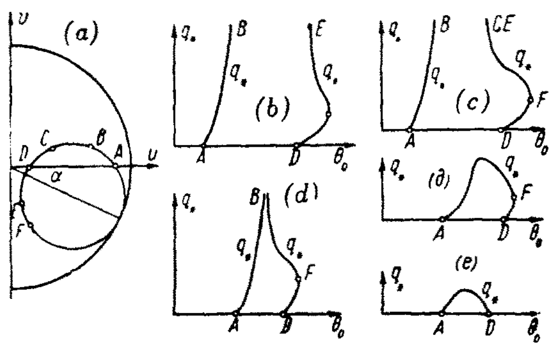

Fig. 10.

At points B and C the magnitude of γ is zero*. On arc AB the

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* Note that when $\kappa > 3$ everywhere on the strophoid $\gamma < 0$, i.e. points B and C do not exist. This leads to the position that curve $q_*(\theta_0)$ is closed for any angle of attack (see below).

magnitude $\gamma_0 < 0$, i.e. the representative point, lies in region *2* where for any θ_0 there exists a unique value of $q = q_{*+}$ at which on the surface of the body the condition of impermeability is fulfilled for $u_{1+} > \sqrt{[(\kappa - 1)/(3\kappa + 1)]}$ (Section 7). When $q < q_{*+}$ then solution II is fulfilled. At point *B* the magnitude $q_{*+} = \infty$ and the solution for $\eta \to 0$ takes the form (7.6).

On the arc *BC*, where $\gamma > 0$ point V_0 lies in regions *3* or *4*. Here for fixed θ_0 and q varying from zero to infinity, solution II is satisfied. If at point *C* the derivative $dv/du > k_\gamma$ then with counter-clockwise motion along the strophoid from *C* point *E* will be met at which the integral curve (6.2) for $q = \infty$ touches line γ.

For arc *CE*, the range in variation of q for which the integral curves come to the origin of coordinates, is not limited from above. For each point on the arc *EF* (point *F* is determined below) it is possible to point out some point $q = q_{*+}$ at which solution I holds, whilst the integral curves approach the u-axis from above and the velocity on the surface of the wedge $u_{1+} > \sqrt{[(\kappa - 1)/(3\kappa + 1)]}$. When $q < q_{*+}$ integral curves II come to the origin whilst, when $q > q_{*+}$ there will be no solution to the boundary value problem.

For each point on the arc *DF* there exists some point $q = q_{*-}$ at which the integral curves I approach the u-axis from below and the velocity on the surface of the wedge is $u_{1-} < \sqrt{[(\kappa - 1)/(3\kappa + 1)]}$. When $q > q_{*-}$ integral curves II approach the origin, whilst when $q < q_{*-}$ the solution to the boundary problem does not exist. Thus for points lying in region *6* on the arc *EF*, the range of variation in q over which there exists some solution to II is determined by the inequalities $q_{*-} < q < q_{*+}$.

When V_0 moves counter-clockwise the quantities u_{1-} and u_{1+} approach each other and at the same time q_{*-} and q_{*+} approach until finally $u_{1-} = u_{1+} = \sqrt{[(\kappa - 1)/(3\kappa + 1)]}$, $q_{*-} = q_{*+}$ at point *F*. If V_0 lies on a strophoid below point *F* there will not be any solution to the boundary value problem.

The character of the relationships between quantities q_{*+} and q_{*-} will now be explained. These quantities present solution I in terms of the angle of inclination of the shock θ_0 for various values of α. When $\alpha = 0$ points *A* and *B* coincide in the plane uv with the point $u = 1$, $v = 0$. This means that the relationship $q_{*+}(\theta_0)$ (the curve passing through *A*) decays into the straight line $\theta_0 = 0$. Calculation has shown that point *E* is located on the arc *DF* when $\alpha = 0$. When $\alpha = 0$ at point *D*, $\theta_0 = \pi/2$ and because of this for small values of α solutions can be obtained (curve passing through point *D*), when $\theta_0 > \pi/2$. On Fig. 10, *b* curves are shown in diagram form for $q_*(\theta_0)$ for small values of α. Each point in the region between the two curves $q_*(\theta_0)$ corresponds to solution II. Observe that the angle of inclination of the tangent to the curve $q_{*+}(\theta_0)$ at point A is determined approximately from the first Equation (8.2).

When α increases the relative location of points (Fig. 10*a*) changes. Points A and D approach each other. Point E moves toward C, goes over to the arc CD and at a certain value of α coincides with C (Fig. 10*b*). This takes place when at point C the derivative $dv/du = k_\gamma$ ($\alpha = 23^\circ.18$ and $\theta_0 = 44^\circ.2$ for $\kappa = 1.4$).

At high values of α the quantity $q = q_*(\theta_0)$ (the curve passing through point D) goes to infinity at point C, and it can be determined by analysis of the initial data without any numerical integration of the equation.

Further increase in α brings us to the point when B and C coincide, as is shown on Fig. 10, *e* ($\alpha = 28^\circ.15$ for $\kappa = 1.4$).

Furthermore the curve $q_*(\theta_0)$ is closed (Fig. 10*e*). In this case point F coincides with D. This takes place when the velocity behind the strong shock $u_0 = \sqrt{[(\kappa - 1)/(3\kappa + 1)]}$. The curve $q_*(\theta_0)$ has a vertical tangent at point D. Then $q_*(\theta_0)$ becomes of the form (Fig. 10), and this corresponds to the case when in region *6* there is no solution. Finally curve $q_*(\theta_0)$ is drawn out to a point and this takes place when α becomes equal to α° – the limiting angle of rotation when there is no field ($\alpha^\circ = 45^\circ.35$ for $\kappa = 1.4$).

Figure 11 shows calculated results for values q_* for $\alpha = 30^\circ$ when curve $q_*(\theta_0)$ is closed (calculated points shown by circles).

12. It follows from what has been said that the case when $\alpha < \alpha^\circ$, for given values of q, two solutions I exist of the boundary problem and they transform into the nonmagnetic solutions, when the intensity of the magnetic field tends to zero and there is an infinite number of solutions II for which the angle of inclination of the shock wave lies within the interval between the two values obtained from solution I. The quantity q, for which there exists a solution with an attached shock wave, can change over the interval between zero to infinity for $0 < \alpha < \alpha_x$ and it will be limited from above for $\alpha_x < \alpha < \alpha^\circ$ ($\alpha_x = 28^\circ.15$ for $\kappa = 1.4$. When κ increases the quantity α_x decreases. With $\kappa \geqslant 3$ the range within which q can be infinitely great is absent).

When there is a solution (6.2) for $q = \infty$, the magnetic field vectors and the velocity vectors are parallel over the whole stream, whilst pressure and density on the surface of the wedge become zero (7.6) (a special type of pinch). As $\alpha (\alpha > \alpha_x)$ approaches α° the range of variation of q where there exists a solution with a connected shock wave decreases, whilst for $\alpha \to \alpha^0$ it draws out to zero.

Although the solution to the boundary value problem turns out to be many valued, it is natural to assume that at least in free flight a solution is realized which transforms into the well known solution for flow round a wedge with weak shock $q \to 0$. This follows from the requirement of a continuous relationship in the solution and the intensity of the magnetic field. For this solution for fixed α, as the intensity of the magnetic field increases from zero the angle of inclination of the shock increases from the nonmagnetic θ_{00} to its limiting value θ_*. When $\alpha < \alpha_x$ the value of θ_* is obtained for infinitely high field intensity, whilst with $\alpha_x < \alpha < \alpha^0$, finite intensity is obtained. In the second case there is no solution with an attached shock when the field intensity is increased further, and this demonstrates that it is possible to transform at a fairly high value of magnetic field from flow with attached shock to flows with detached shock.

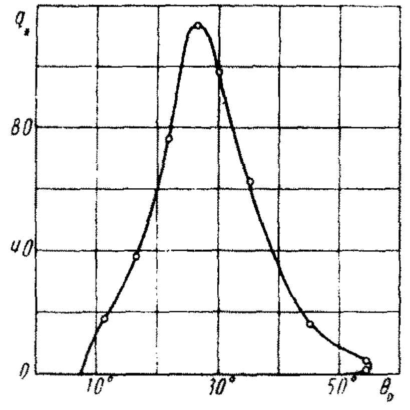

Fig. 11.

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